S/26

Mathematics Data Book

for Part I of the Engineering Tripos

2002 Edition

Cambridge University Engineering Department

1. Complex Variables

General

$z = x + iy = r(\cos\theta + i\sin\theta) = re^{i\theta}$ where $i^2 = -1$ and $-\pi < \theta \le \pi$

Real part $\mathrm{Re}(z) = x$ Imaginary part $\mathrm{Im}(z) = y$

For integer n,

$$e^{2n\pi i} = 1 \qquad z = re^{i(\theta + 2n\pi)}$$

$$z^\alpha = r^\alpha \exp[i\alpha(\theta + 2n\pi)]$$

$$\ln z = \ln r + i(\theta + 2n\pi)$$

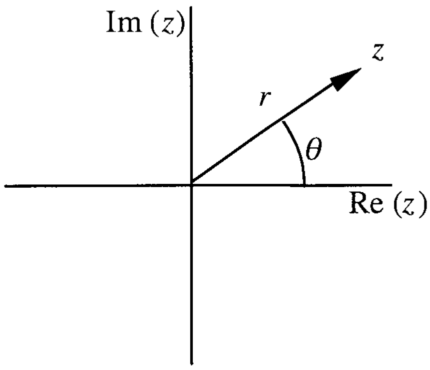
Complex conjugate

$\bar{z} = x - iy = re^{-i\theta}$; $z\bar{z} = |z|^2$ which is purely real

(z^* also used to denote complex conjugate)

$$\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2 \qquad \overline{z_1 z_2} = \bar{z}_1\bar{z}_2 \qquad \overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$$

Argand diagram



$$r = |z|$$
$$\theta = \arg(z)$$

De Moivre's Theorem

$$(\cos\theta + i\sin\theta)^n = e^{in\theta} = \cos n\theta + i\sin n\theta$$

2. Limits

$n^s x^n \to 0$ as $n \to \infty$ if $|x| < 1$ for any real value of s

$\dfrac{x^n}{n!} \to 0$ as $n \to \infty$

$\left(1 + \dfrac{x}{n}\right)^n \to e^x$ as $n \to \infty$

$x^s \ln x \to 0$ as $x \to 0$ where $s > 0$.

3. Trigonometric Functions

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i} \qquad \cos x = \frac{e^{ix} + e^{-ix}}{2}$$

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B \qquad \cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

$$\sin A + \sin B = 2 \sin \frac{A+B}{2} \cos \frac{A-B}{2} \qquad \sin A - \sin B = 2 \cos \frac{A+B}{2} \sin \frac{A-B}{2}$$

$$\cos A + \cos B = 2 \cos \frac{A+B}{2} \cos \frac{A-B}{2} \qquad \cos A - \cos B = -2 \sin \frac{A+B}{2} \sin \frac{A-B}{2}$$

$$\sin A \ \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)] \qquad \cos A \ \cos B = \frac{1}{2}[\cos(A+B) + \cos(A-B)]$$

$$\sin A \ \cos B = \frac{1}{2}[\sin(A+B) + \sin(A-B)]$$

$$\sin^2 x = \frac{1}{2}[1 - \cos 2x] \qquad \cos^2 x = \frac{1}{2}[1 + \cos 2x]$$

$$\sin^3 x = \frac{1}{4}[3 \sin x - \sin 3x] \qquad \cos^3 x = \frac{1}{4}[3 \cos x + \cos 3x]$$

4. Hyperbolic Functions

$$\cosh x = \frac{e^{x} + e^{-x}}{2} \qquad \sinh x = \frac{e^{x} - e^{-x}}{2}$$

$$\cosh ix = \cos x \qquad \cos ix = \cosh x$$

$$\sinh ix = i \sin x \qquad \sin ix = i \sinh x$$

$$\cosh^2 x - \sinh^2 x = 1$$

$$\cosh(x \pm y) = \cosh x \ \cosh y \pm \sinh x \sinh y$$

$$\sinh(x \pm y) = \sinh x \ \cosh y \pm \cosh x \sinh y$$

$$\cosh(x \pm iy) = \cosh x \ \cos y \pm i \sinh x \sin y$$

$$\sinh(x \pm iy) = \sinh x \ \cos y \pm i \cosh x \sin y$$

5. Error Function (Gaussian Integral)

$$\operatorname{erf} x = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-u^2)\, du$$

x	0	.25	.5	.75	1	1.25	1.5	2	∞
erf x	0	.276	.520	.711	.843	.923	.966	.995	1

For $\lambda > 0$, $\displaystyle\int_0^\infty \exp(-\lambda u^2)\, du = \left[\frac{\pi}{4\lambda}\right]^{\frac{1}{2}}$ $\qquad \displaystyle\int_{-\infty}^\infty \exp(-\lambda u^2)\, du = \left[\frac{\pi}{\lambda}\right]^{\frac{1}{2}}$

6. Series

Arithmetic

$$S_n = a + (a+d) + (a+2d) + \ldots + (a + (n-1)d) \;=\; \frac{n}{2}\,[2a + (n-1)d\,]$$

Geometric

$$S_n = a + ar + ar^2 + \ldots + ar^{n-1} \;=\; \frac{a(1-r^n)}{1-r}$$

$$S_\infty = \frac{a}{1-r} \qquad \text{provided } |r| < 1$$

Binomial expansion

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots$$

If n is a positive integer the series terminates and is valid for all x. The general term is then ${}^nC_r\,x^r$, also written $\binom{n}{r}x^r$, where ${}^nC_r = \dfrac{n!}{r!(n-r)!}$

When n is not a positive integer, the series does not terminate; the resulting infinite series is convergent for $|x| < 1$.

Taylor series

For a function of a single variable (real or complex)

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \ldots. + \frac{h^n}{n!}f^{(n)}(x) + \ldots$$

(When $x = 0$ this is often called a **McLaurin** series)

For two variables

$$f(x+h,\, y+k) = f(x,y) + \left[h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}\right] + \frac{1}{2!}\left[h^2\frac{\partial^2 f}{\partial x^2} + 2hk\frac{\partial^2 f}{\partial x \partial y} + k^2\frac{\partial^2 f}{\partial y^2}\right] + \ldots$$

in which subsequent square brackets involve the binomial coefficients (1,3,3,1), (1,4,6,4,1), etc and all the derivatives are evaluated at (x,y).

Integer series

$$\sum_{1}^{N} n = 1 + 2 + 3 + ... + N = \frac{1}{2} N(N+1)$$

$$\sum_{1}^{N} n^2 = 1^2 + 2^2 + 3^2 + ... + N^2 = \frac{1}{6} N(N+1)(2N+1)$$

$$\sum_{1}^{N} n^3 = 1^3 + 2^3 + 3^3 + ... + N^3 = [1 + 2 + 3 + ... + N]^2 = \frac{1}{4} N^2 (N+1)^2$$

$$\sum_{1}^{N} n(n+1)(n+2)....(n+r) = \frac{N(N+1)(N+2).....(N+r)(N+r+1)}{(r+2)}$$

$$\sum_{1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + ... = \ln 2$$ (See expansion of $\ln(1+z)$)

$$\sum_{1}^{\infty} \frac{(-1)^{n+1}}{2n-1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + ... = \frac{\pi}{4}$$ (See expansion of $\tan^{-1} z$)

$$\sum_{1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + ... = \frac{\pi^2}{6}$$

Power series (Valid for real and complex numbers)

$$e^z = 1 + z + \frac{z^2}{2!} + ... + \frac{z^n}{n!} ....$$ convergent for all z

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - ....$$ convergent for all z

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - ....$$ convergent for all z

$$\sinh z = z + \frac{z^3}{3!} + \frac{z^5}{5!} + ...$$ convergent for all z

$$\cosh z = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + ...$$ convergent for all z

$$\tan z = z + \frac{1}{3} z^3 + \frac{2}{15} z^5 + \frac{17}{315} z^7 ...$$ convergent for $|z| < \frac{\pi}{2}$

$$\sin^{-1} z = z + \frac{1}{2.3} z^3 + \frac{1.3\, z^5}{2.4\ 5} + ...$$ convergent for $|z| < 1$

$$\tan^{-1} z = z - \frac{z^3}{3} + \frac{z^5}{5} .....$$ convergent both on and within circle $|z| = 1$ except at the point $z = \pm i$

$$\ln(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - ......$$ Principal Value of $\ln(1+z)$ converges both on and within circle $|z| = 1$ except at the point $z = -1$

7. Differentiation

For vectors and scalars which are functions of a single variable

$$(uv)' = u'v + uv' \qquad \left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$$

$$(\mathbf{a}.\mathbf{b})' = \mathbf{a}'.\mathbf{b} + \mathbf{a}.\mathbf{b}' \qquad (\mathbf{a}\times\mathbf{b})' = \mathbf{a}'\times\mathbf{b} + \mathbf{a}\times\mathbf{b}'$$

$$(u\mathbf{a})' = u'\mathbf{a} + u\mathbf{a}'$$

Leibniz Theorem

$$(uv)^{(n)} = u^{(n)}v + n\,u^{(n-1)}v' + \ldots + {}^nC_p\, u^{(n-p)}v^{(p)} + \ldots + uv^{(n)}$$

where $ {}^nC_p \equiv \binom{n}{p} = \frac{n!}{p!(n-p)!}$

8. Partial Differentiation

Stationary points

A function $\phi(x,y)$ has a stationary point when $\frac{\partial\phi}{\partial x} = \frac{\partial\phi}{\partial y} = 0$.

Provided Δ is non-zero at a stationary point, where $\Delta = \frac{\partial^2\phi}{\partial x^2}\frac{\partial^2\phi}{\partial y^2} - \left[\frac{\partial^2\phi}{\partial x\partial y}\right]^2$, the following conditions on the second derivatives there determine whether it is a maximum, a minimum or a saddle point,

Maximum: $\Delta > 0$, $\frac{\partial^2\phi}{\partial x^2} < 0$, and $\frac{\partial^2\phi}{\partial y^2} < 0$

Minimum: $\Delta > 0$, $\frac{\partial^2\phi}{\partial x^2} > 0$, and $\frac{\partial^2\phi}{\partial y^2} > 0$

Saddle point: all other cases for which Δ is non-zero.

The case $\Delta = 0$ can be a maximum, a minimum, a saddle point, or none of these.

Total differential theorem

For a function $\phi(x,y,z,\ldots)$

$$d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz + \ldots$$

in which $\frac{\partial\phi}{\partial x}$ means $\left(\frac{\partial\phi}{\partial x}\right)_{y,z\ldots}$ (i.e. with $y, z \ldots$ kept constant).

Chain rule

When $x,y,z\ldots$ are functions of $u,v,w\ldots$

$$\left(\frac{\partial\phi}{\partial u}\right)_{v,w\ldots} = \frac{\partial\phi}{\partial x}\left(\frac{\partial x}{\partial u}\right)_{v,w\ldots} + \frac{\partial\phi}{\partial y}\left(\frac{\partial y}{\partial u}\right)_{v,w\ldots} + \frac{\partial\phi}{\partial z}\left(\frac{\partial z}{\partial u}\right)_{v,w\ldots} + \ldots\ldots$$

9. Differential Equations

Integrating factor

A first order o.d.e. of the form

$$\frac{dy}{dx} + P(x)\, y = Q(x)$$

can be integrated using the <u>integrating factor</u> $\exp(\int P\, dx)$, so that the equation takes the form

$$\frac{d}{dx}\left[y \exp(\textstyle\int P\, dx')\right] = Q(x) \exp(\textstyle\int P\, dx')$$

Particular integrals

For linear differential equations with constant coefficients:

Right-hand side	Trial P.I.
constant	a
x^n (n integer)	$a\, x^n + b\, x^{n-1} + \ldots$
e^{kx}	$a\, e^{kx}$
$x\, e^{kx}$	$(a\, x + b)\, e^{kx}$
$x^n\, e^{kx}$	$(a\, x^n + b\, x^{n-1} + \ldots)\, e^{kx}$
$\left.\begin{matrix}\sin px \\ \cos px\end{matrix}\right\}$	$a \sin px + b \cos px$
$\left.\begin{matrix}e^{kx} \sin px \\ e^{kx} \cos px\end{matrix}\right\}$	$e^{kx} (a \sin px + b \cos px)$

For the special case when the right hand side has an exponential or trigonometric factor which is also a solution of the differential equation:

Complementary Function	Right-hand side	Trial P.I.
e^{kx}	e^{kx}	$a\, x\, e^{kx}$
e^{kx}	$x^n\, e^{kx}$	$(a\, x^{n+1} + b\, x^n + \ldots)\, e^{kx}$
$\left.\begin{matrix}\sin px \\ \cos px\end{matrix}\right\}$	$\left.\begin{matrix}\sin px \\ \cos px\end{matrix}\right\}$	$x\, (a \sin px + b \cos px)$
$\left.\begin{matrix}e^{kx} \sin px \\ e^{kx} \cos px\end{matrix}\right\}$	$\left.\begin{matrix}e^{kx} \sin px \\ e^{kx} \cos px\end{matrix}\right\}$	$x\, e^{kx} (a \sin px + b \cos px)$

10. Integration

Standard indefinite integrals

Integrand	Integral	Integrand	Integral
$\sin x$	$-\cos x$	$\sinh x$	$\cosh x$
$\cos x$	$\sin x$	$\cosh x$	$\sinh x$
$\tan x$	$-\ln(\cos x)$	$\tanh x$	$\ln(\cosh x)$
$\operatorname{cosec} x$	$\ln(\tan \frac{x}{2})$	$\operatorname{cosech} x$	$\ln(\tanh \frac{x}{2})$
$\sec x$	$\ln(\tan x + \sec x)$	$\operatorname{sech} x$	$2\tan^{-1}(e^x)$
$\cot x$	$\ln(\sin x)$	$\coth x$	$\ln(\sinh x)$
$\sec^2 x$	$\tan x$	$\operatorname{sech}^2 x$	$\tanh x$
$\tan x \ \sec x$	$\sec x$	$\tanh x \operatorname{sech} x$	$-\operatorname{sech} x$
$\cot x \ \operatorname{cosec} x$	$-\operatorname{cosec} x$	$\coth x \operatorname{cosech} x$	$-\operatorname{cosech} x$

$\dfrac{1}{\sqrt{a^2 - x^2}}$	$\sin^{-1}\left(\dfrac{x}{a}\right)$	or	$-\cos^{-1}\left(\dfrac{x}{a}\right)$
$\dfrac{1}{\sqrt{x^2 + a^2}}$	$\sinh^{-1}\left(\dfrac{x}{a}\right)$	or	$\ln(x + \sqrt{x^2 + a^2})$
$\dfrac{1}{\sqrt{x^2 - a^2}}$	$\cosh^{-1}\left(\dfrac{x}{a}\right)$	or	$\ln(x + \sqrt{x^2 - a^2})$
$\dfrac{1}{x^2 + a^2}$	$\dfrac{1}{a}\tan^{-1}\left(\dfrac{x}{a}\right)$		

Standard substitutions

If the integrand is a function of:			Substitute:		
$(a^2 - x^2)$	or	$\sqrt{a^2 - x^2}$	$x = a\ \sin\theta$	or	$x = a\ \cos\theta$
$(a^2 + x^2)$	or	$\sqrt{a^2 + x^2}$	$x = a\ \tan\theta$	or	$x = a\ \sinh\theta$
$(x^2 - a^2)$	or	$\sqrt{x^2 - a^2}$	$x = a\ \sec\theta$	or	$x = a\ \cosh\theta$

or of the form: $\dfrac{1}{(ax+b)\sqrt{px+q}}$ $\qquad px + q = u^2$

$\dfrac{1}{(ax+b)\sqrt{px^2+qx+r}}$ $\qquad ax + b = \dfrac{1}{u}$

or a rational function of $\sin x$ and/or $\cos x$ $\qquad t = \tan \dfrac{x}{2}$

[whence $\quad \sin x = \dfrac{2t}{1+t^2} \qquad \cos x = \dfrac{1-t^2}{1+t^2} \qquad dx = \dfrac{2dt}{1+t^2}$]

Integration by parts

$$\int_a^b u\left(\frac{dv}{dx}\right)dx = [uv]_a^b - \int_a^b v\left(\frac{du}{dx}\right)dx$$

Differentiation of an integral

$$\frac{d}{dx}\int_{a(x)}^{b(x)} f(x,y)\,dy = f(x,b)\,\frac{db}{dx} - f(x,a)\,\frac{da}{dx} + \int_{a(x)}^{b(x)}\frac{\partial f(x,y)}{\partial x}dy$$

Change of variable in surface and volume integration

Surface:

$\iint_S f(x,y)\; dxdy \;=\; \iint_{S'} F(u,v)\;|J|\; dudv$ where $u(x,y)$ and $v(x,y)$ are the new variables

and where $J \equiv \dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix}$ is the Jacobian.

For surface integrals involving vector normals

$$\mathbf{n}\,dA \;\equiv\; \mathbf{n}\,dxdy \;=\; \pm\,\frac{\partial \mathbf{r}}{\partial u}\times\frac{\partial \mathbf{r}}{\partial v}\;dudv$$

and the sign is chosen to preserve the sense of the normal.

Volume

$$\iiint_V f(x,y,z)\; dxdydz \;=\; \iiint_{V'} F(u,v,w)\;|J|\; dudvdw$$

where $J \equiv \dfrac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} & \dfrac{\partial x}{\partial w} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} & \dfrac{\partial y}{\partial w} \\ \dfrac{\partial z}{\partial u} & \dfrac{\partial z}{\partial v} & \dfrac{\partial z}{\partial w} \end{vmatrix}$

Note

$$\frac{1}{J} = \frac{\partial(u,v,...)}{\partial(x,y,...)}$$

11. Vector Products

Scalar Product

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\ |\mathbf{b}| \cos\theta \qquad \text{(where } \theta \text{ is the angle between the vectors)}$$

$$= \mathbf{a}^t \mathbf{b} = \mathbf{b} \cdot \mathbf{a} = \mathbf{b}^t \mathbf{a} = a_x b_x + a_y b_y + a_z b_z = [\, a_x \ a_y \ a_z \,] \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix}$$

Vector Product

$$\mathbf{a} \times \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}| \sin\theta \ \mathbf{n}$$

(where θ is the angle between the vectors, and **n** is a unit vector normal to the plane containing **a** and **b** such that **a**, **b** , **n** form a right-handed set)

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} = -\mathbf{b} \times \mathbf{a}$$

Scalar Triple Product

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix} = \begin{vmatrix} a_x & b_x & c_x \\ a_y & b_y & c_y \\ a_z & b_z & c_z \end{vmatrix}$$

$$= \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})$$

$$= -\mathbf{a} \cdot (\mathbf{c} \times \mathbf{b}) = -\mathbf{c} \cdot (\mathbf{b} \times \mathbf{a}) = -\mathbf{b} \cdot (\mathbf{a} \times \mathbf{c})$$

The notation [**a**, **b**, **c**] is also used for $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$.

Vector Triple Product

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\,\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\,\mathbf{c}$$

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = (\mathbf{a} \cdot \mathbf{c})\,\mathbf{b} - (\mathbf{b} \cdot \mathbf{c})\,\mathbf{a}$$

12. Matrices and Linear Algebra

$(\mathbf{AB...N})^t$	=	$\mathbf{N}^t...\mathbf{B}^t\mathbf{A}^t$	where $(\,.\,)^t$ denotes the transpose
$(\mathbf{AB...N})^{-1}$	=	$\mathbf{N}^{-1}...\mathbf{B}^{-1}\mathbf{A}^{-1}$	(if individual inverses exist)
$\det(\mathbf{AB...N})$	=	$\det \mathbf{A} \, \det \mathbf{B} \, ...\det \mathbf{N}$	(if individual matrices are square)

If $\mathbf{A}$ is square and if $\mathbf{A}^{-1}$ exists (i.e. if $\det \mathbf{A} \neq 0$), then $\mathbf{Ax} = \mathbf{b}$ has a unique solution

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$$

If $\mathbf{A}$ is square then $\mathbf{Ax} = 0$ has a non-trivial solution if and only if $\det \mathbf{A} = 0$.

For an orthogonal matrix

$$\mathbf{Q}^{-1} = \mathbf{Q}^t, \quad \det \mathbf{Q} = \pm 1.$$

$\mathbf{Q}^t$ is also orthogonal.

If $\det \mathbf{Q} = +1$ then $\mathbf{Q}$ describes a rotation without reflection.

Eigenvalues and Eigenvectors

If $\mathbf{A}$ is an $n \times n$ matrix, its eigenvalues λ and corresponding eigenvectors $\mathbf{u}$ satisfy

$$\mathbf{Au} = \lambda\mathbf{u}.$$

There are in general n eigenvalues λ_i and corresponding eigenvectors $\mathbf{u}_i$.

The eigenvalues are the roots of the n'th order polynomial equation

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0$$

where $\mathbf{I}$ is the identity matrix.

If $\mathbf{A}$ is real and symmetric the eigenvalues are real and the eigenvectors corresponding to different eigenvalues are orthogonal. For repeated eigenvalues, the corresponding eigenvectors can be chosen to be orthogonal. Furthermore,

$$\mathbf{U}^t\mathbf{AU} = \Lambda \quad \text{and} \quad \mathbf{A} = \mathbf{U}\Lambda\mathbf{U}^t$$

where Λ is the diagonal matrix whose elements are the eigenvalues of $\mathbf{A}$ and $\mathbf{U}$ is the orthogonal matrix whose columns are the normalized eigenvectors of $\mathbf{A}$.

Rayleigh's quotient

If $\mathbf{x}$ is an approximation to an eigenvector of $\mathbf{A}$ then $\dfrac{\mathbf{x}^t\mathbf{Ax}}{\mathbf{x}^t\mathbf{x}}$ is a good approximation to the corresponding eigenvalue.

Material relevant to IB Linear Algebra

Rank

The rank, r, of a matrix is the number of independent rows, or columns.

Fundamental Subspaces of an $m \times n$ matrix A

The *column space* is the space spanned by the columns. It has dimension equal to the rank, r, and is a subspace of R^m.

The *nullspace* is the space spanned by the solutions $\mathbf{x}$ of the equation $\mathbf{Ax} = \mathbf{0}$. The nullspace has dimension $n - r$ and is a subspace of R^n.

The *row space* is the space spanned by the rows of $\mathbf{A}$. It has dimension equal to r and is a subspace of R^n.

The *left-nullspace* is the space spanned by the solutions $\mathbf{y}$ of the equation $\mathbf{y}^t \mathbf{A} = \mathbf{0}$. It has dimension $m - r$, and is a subspace of R^m.

The nullspace is the orthogonal complement of the row space in R^n.
The left-nullspace is the orthogonal complement of the column space in R^m.

For $\mathbf{A}\,\mathbf{x} = \mathbf{b}$ to have a solution, $\mathbf{b}$ must lie in the column space, i.e. $\mathbf{y}^t\,\mathbf{b} = \mathbf{0}$ for any $\mathbf{y}$ such that $\mathbf{A}^t\,\mathbf{y} = \mathbf{0}$.

Decompositions of an $m \times n$ matrix A

LU Decomposition

$\mathbf{PA} = \mathbf{LU}$, where $\mathbf{P}$ is a permutation matrix, $\mathbf{L}$ a lower triangular matrix and $\mathbf{U}$ an $m \times n$ echelon matrix.

QR Decomposition

$\mathbf{A} = \mathbf{QR}$, where the columns of $\mathbf{Q}$ are orthonormal, and $\mathbf{R}$ is upper-triangular and invertible. When $m = n$ and so all matrices are square, $\mathbf{Q}$ is an orthogonal matrix.

Eigenvalue Decomposition (only for $m = n$)

Provided that $\mathbf{A}$ has *n linearly independent eigenvectors,* $\mathbf{A} = \mathbf{S}\,\mathbf{\Lambda}\,\mathbf{S}^{-1}$, where $\mathbf{S}$ has the eigenvectors of $\mathbf{A}$ as its columns, and $\mathbf{\Lambda}$ is a diagonal matrix with eigenvalues along the diagonal.

If $\mathbf{A}$ is real and symmetric, see under Eigenvalues and Eigenvectors above.

Singular Value Decomposition

$$\mathbf{A} = \mathbf{Q_1}\, \Sigma\, \mathbf{Q_2^t} \quad \text{(orthogonal} \times \text{diagonal} \times \text{orthogonal)}$$

- The columns of $\mathbf{Q_1}$ $(m \times m)$ are the eigenvectors of $\mathbf{A}\mathbf{A}^t$
- The columns of $\mathbf{Q_2}$ $(n \times n)$ are the eigenvectors of $\mathbf{A}^t\mathbf{A}$
- The r singular values, arranged in descending order on the diagonal of $\mathbf{\Sigma}$ $(m \times n)$ are the square roots of the non-zero eigenvalues of both $\mathbf{A}\mathbf{A}^t$ and $\mathbf{A}^t\mathbf{A}$. r is the rank of the matrix.

Basis of column space:	first r columns of $\mathbf{Q_1}$
Basis of left nullspace:	last $m - r$ columns of $\mathbf{Q_1}$
Basis of row space:	first r columns of $\mathbf{Q_2}$
Basis of nullspace:	last $n - r$ columns of $\mathbf{Q_2}$.

General solution of $\mathbf{A}\mathbf{x} = \mathbf{b}$ by Gaussian Elimination

1. Transform $\mathbf{A}\mathbf{x} = \mathbf{b}$ into $\mathbf{U}\mathbf{x} = \mathbf{c}$.
2. Set all free variables to zero and find a particular solution $\mathbf{x}_0$.
3. Set the RHS to zero, give each free variable in turn the value 1 while the others are zero, and solve to find a set of vectors which span the nullspace of **A**. Arrange these vectors as the columns of a matrix $\mathbf{X}$.
4. The general solution is $\mathbf{x}_0 + \mathbf{X}\alpha$, where α is arbitrary.

Least squares solution of $\mathbf{A}\mathbf{x} = \mathbf{b}$ using QR

Solve $\mathbf{R}\bar{\mathbf{x}} = \mathbf{Q}^t\mathbf{b}$ by back-substitution.

13. Vector Calculus

ϕ is a scalar function of position and $\mathbf{u}$ a vector function.

Cartesian coordinates x, y, z ; $\mathbf{u} = u_x \mathbf{i} + u_y \mathbf{j} + u_z \mathbf{k}$

$$\text{grad } \phi \equiv \nabla \phi \equiv \mathbf{i}\frac{\partial \phi}{\partial x} + \mathbf{j}\frac{\partial \phi}{\partial y} + \mathbf{k}\frac{\partial \phi}{\partial z}$$

$$\text{div } \mathbf{u} \equiv \nabla . \mathbf{u} \equiv \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}$$

$$\text{curl } \mathbf{u} \equiv \nabla \times \mathbf{u} \equiv \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ u_x & u_y & u_z \end{vmatrix} \equiv \begin{bmatrix} 0 & -\partial/\partial z & \partial/\partial y \\ \partial/\partial z & 0 & -\partial/\partial x \\ -\partial/\partial y & \partial/\partial x & 0 \end{bmatrix} \begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix}$$

$$\text{div (grad } \phi) \equiv \nabla^2 \phi \equiv \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} \qquad \text{(the Laplacian operator)}$$

Cylindrical polar coordinates r, θ, z ; $\mathbf{u} = u_r \mathbf{e}_r + u_\theta \mathbf{e}_\theta + u_z \mathbf{e}_z$

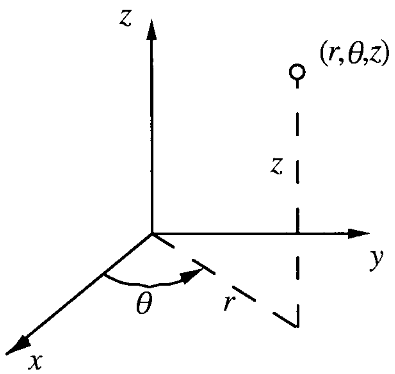
($\mathbf{e}_r$, $\mathbf{e}_\theta$ and $\mathbf{e}_z$ are unit radial, tangential and axial vectors respectively)

$$\text{grad } \phi \equiv \nabla \phi \equiv \mathbf{e}_r \frac{\partial \phi}{\partial r} + \frac{\mathbf{e}_\theta}{r}\frac{\partial \phi}{\partial \theta} + \mathbf{e}_z \frac{\partial \phi}{\partial z}$$

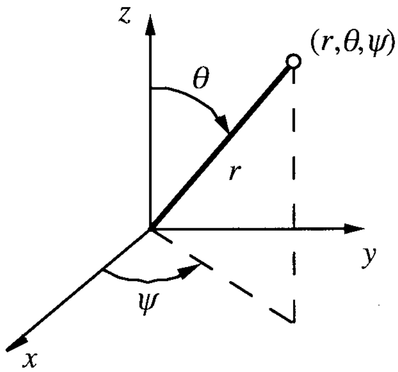
$$\text{div } \mathbf{u} \equiv \nabla . \mathbf{u} \equiv \frac{1}{r}\frac{\partial (r u_r)}{\partial r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z}$$

$$\text{curl } \mathbf{u} \equiv \nabla \times \mathbf{u} \equiv \frac{1}{r}\begin{vmatrix} \mathbf{e}_r & r\mathbf{e}_\theta & \mathbf{e}_z \\ \partial/\partial r & \partial/\partial \theta & \partial/\partial z \\ u_r & r u_\theta & u_z \end{vmatrix}$$

$$\text{div (grad } \phi) \equiv \nabla^2 \phi \equiv \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial \phi}{\partial r} \right) + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} + \frac{\partial^2 \phi}{\partial z^2}$$



Spherical polar coordinates r, θ, ψ ; $\mathbf{u} = u_r \mathbf{e}_r + u_\theta \mathbf{e}_\theta + u_\psi \mathbf{e}_\psi$ where $0 \le \theta \le \pi$; $0 \le \psi \le 2\pi$
($\mathbf{e}_r$, $\mathbf{e}_\theta$ and $\mathbf{e}_\psi$ are unit radial, longitudinal and azimuthal vectors respectively)



$$\text{grad } \phi \equiv \nabla \phi \equiv \mathbf{e}_r \frac{\partial \phi}{\partial r} + \frac{\mathbf{e}_\theta}{r}\frac{\partial \phi}{\partial \theta} + \frac{\mathbf{e}_\psi}{r\sin\theta}\frac{\partial \phi}{\partial \psi}$$

$$\text{div } \mathbf{u} \equiv \nabla . \mathbf{u} \equiv \frac{1}{r^2}\frac{\partial (r^2 u_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\sin\theta\, u_\theta)}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial u_\psi}{\partial \psi}$$

$$\text{curl } \mathbf{u} \equiv \nabla \times \mathbf{u} \equiv \frac{1}{r^2 \sin\theta}\begin{vmatrix} \mathbf{e}_r & r\mathbf{e}_\theta & r\sin\theta\,\mathbf{e}_\psi \\ \partial/\partial r & \partial/\partial\theta & \partial/\partial\psi \\ u_r & ru_\theta & r\sin\theta\, u_\psi \end{vmatrix}$$

$$\text{div (grad } \phi) \equiv \nabla^2 \phi \equiv \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \phi}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \phi}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 \phi}{\partial \psi^2}$$

Spherical symmetry $\phi = \phi(r)$, $\mathbf{u} = u(r)\,\mathbf{e}_r$ ($\mathbf{e}_r$ is a unit radial vector)

$$\text{grad } \phi \equiv \mathbf{e}_r \frac{d\phi}{dr}$$

$$\text{div (grad } \phi) \equiv \nabla^2 \phi \equiv \frac{d^2\phi}{dr^2} + \frac{2}{r}\frac{d\phi}{dr} = \frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{d\phi}{dr}\right)$$

$$\text{div } \mathbf{u} \equiv \frac{1}{r^2}\frac{d}{dr}(r^2 u)$$

$$\text{curl } \mathbf{u} \equiv 0$$

Potentials

A vector field $\mathbf{u}$ is said to be *irrotational* if $\nabla \times \mathbf{u} = 0$.

A vector field $\mathbf{u}$ is said to be *solenoidal* or *incompressible* if $\nabla.\mathbf{u} = 0$.

If $\nabla \times \mathbf{u} = 0$, then there exists a scalar potential ϕ such that $\mathbf{u} = \nabla\phi$
(for some applications it is more natural to use $\mathbf{u} = -\nabla\phi$).

If $\nabla.\mathbf{u} = 0$, then there exists a vector potential $\mathbf{A}$ such that $\mathbf{u} = \nabla \times \mathbf{A}$.
($\mathbf{A}$ is usually chosen so that $\nabla.\mathbf{A} = 0$)

Identities

$\nabla(\phi_1 + \phi_2) = \nabla\phi_1 + \nabla\phi_2$ $\qquad$ $\nabla.(\mathbf{u}_1 + \mathbf{u}_2) = \nabla.\mathbf{u}_1 + \nabla.\mathbf{u}_2$

$\nabla \times (\mathbf{u}_1 + \mathbf{u}_2) = \nabla \times \mathbf{u}_1 + \nabla \times \mathbf{u}_2$ $\qquad$ $\nabla.(\phi\mathbf{u}) = \phi\nabla.\mathbf{u} + (\nabla\phi).\mathbf{u}$

$\nabla \times (\phi\mathbf{u}) = \phi\nabla \times \mathbf{u} + (\nabla\phi) \times \mathbf{u}$ $\qquad$ $\nabla.(\mathbf{u}_1 \times \mathbf{u}_2) = \mathbf{u}_2.\nabla \times \mathbf{u}_1 - \mathbf{u}_1.\nabla \times \mathbf{u}_2$

$\nabla.\nabla \times \mathbf{u} = 0$ $\qquad$ $\nabla \times \nabla\phi = 0$

$\nabla \times (\nabla \times \mathbf{u}) = \nabla(\nabla.\mathbf{u}) - \nabla^2\mathbf{u}$ $\quad$ where $\nabla^2\mathbf{u} = (\nabla^2 u_x, \nabla^2 u_y, \nabla^2 u_z)$

$$\mathbf{u} \times (\nabla \times \mathbf{u}) + (\mathbf{u}.\nabla)\mathbf{u} = \nabla\left[\frac{1}{2}\mathbf{u}^2\right]$$

$$\nabla \times (\mathbf{u}_1 \times \mathbf{u}_2) = \mathbf{u}_1\nabla.\mathbf{u}_2 - \mathbf{u}_2\nabla.\mathbf{u}_1 + (\mathbf{u}_2.\nabla)\mathbf{u}_1 - (\mathbf{u}_1.\nabla)\mathbf{u}_2$$

Gauss' Theorem (Divergence Theorem)

$$\iiint_V \nabla.\mathbf{u}\ dV = \iint_S \mathbf{u}.d\mathbf{A}$$

for a <u>closed</u> surface S enclosing a volume V. The <u>outward</u> normal is taken for $d\mathbf{A}$.

Stokes' Theorem

$$\iint_S \nabla \times \mathbf{u}.\,d\mathbf{A} = \oint_C \mathbf{u}.d\mathbf{l}$$

for an <u>open</u> surface S with a closed boundary curve C (the 'rim'). The normal to the surface and the sense of the line integral are related by a <u>right</u> hand screw rule.

14. Fourier Series

Full range

For $-\pi \le \theta \le \pi$

$$f(\theta) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos n\theta + b_n \sin n\theta)$$

where $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\cos n\theta \, d\theta$, $\quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\sin n\theta \, d\theta$

Equivalently $\quad f(\theta) = \sum_{n=-\infty}^{\infty} c_n e^{in\theta}$

where $c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta) e^{-in\theta} d\theta \quad = \frac{1}{2}(a_n - ib_n) \quad$ for $n > 0$

$= \frac{1}{2}(a_{-n} + ib_{-n}) \quad$ for $n < 0$

$= \frac{1}{2}a_0 \quad$ for $n = 0$

If the function $f(\theta)$ is periodic, of period 2π, then these relationships are valid for all θ. The integrals may then be taken over <u>any</u> range of 2π.

Half range

If a Fourier series representation of $f(\theta)$ is required to be valid only in $0 \le \theta \le \pi$, then it need contain either the sine terms alone or the cosine terms alone. For example

$$f(\theta) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos n\theta$$

where $a_n = \frac{2}{\pi}\int_{0}^{\pi} f(\theta)\cos n\theta \, d\theta$

General Range $0 \le t \le T$

$$f(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{2\pi nt}{T} + b_n \sin\frac{2\pi nt}{T}\right)$$

$$\text{where } a_n = \frac{2}{T}\int_0^T f(t)\cos\frac{2\pi nt}{T}dt, \qquad b_n = \frac{2}{T}\int_0^T f(t)\sin\frac{2\pi nt}{T}dt$$

Equivalently $f(t) = \sum_{n=-\infty}^{\infty} c_n e^{i2\pi nt/T}$ where $c_n = \frac{1}{T}\int_0^T f(t)e^{-i2\pi nt/T}\,dt$

i.e. $f(t) = \sum_{n=-\infty}^{\infty} c_n e^{in\omega_0 t}$ where $c_n = \frac{1}{T}\int_0^T f(t)e^{-in\omega_0 t}\,dt$

The (scientific) **fundamental** frequency is $\omega_0 = \frac{2\pi}{T}$ and the (scientific) **n'th harmonic** is $n\omega_0$.

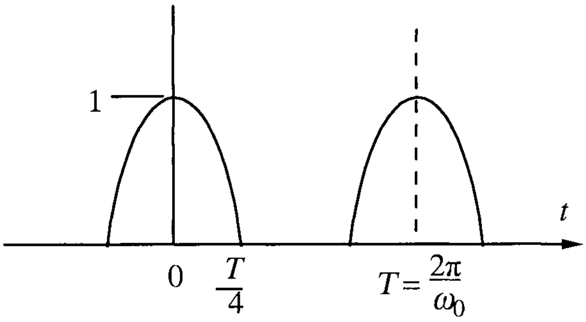
Examples

Some specific complex Fourier series are shown overleaf. Examples of specific real Fourier series can be found in the Electrical Data Book.

15. Fourier Transforms

$$\hat{y}(\omega) = \int_{-\infty}^{\infty} y(t)e^{-i\omega t}\,dt \quad : \quad y(t) = \int_{-\infty}^{\infty} \hat{y}(\omega)e^{i\omega t}\frac{d\omega}{2\pi}$$

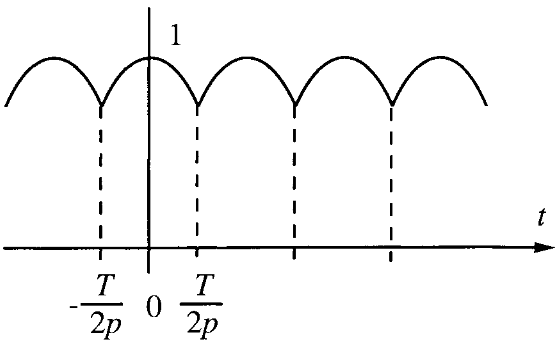
Caution -
(a) Fourier transforms are sometimes written in terms of frequency $f = \omega/2\pi$
(b) Some books handle the 2π factor differently and define transforms with differences in signs of the exponent



Half-wave rectified cosine wave:

$$f(t) = \frac{1}{\pi} + \frac{1}{4}\,e^{i\omega_0 t} + \frac{1}{4}\,e^{-i\omega_0 t} + \frac{1}{\pi} \sum_{\substack{n=-\infty \\ n\ \mathrm{even} \\ n\neq 0}}^{\infty} (\pm 1)\frac{e^{in\omega_0 t}}{n^2-1}$$

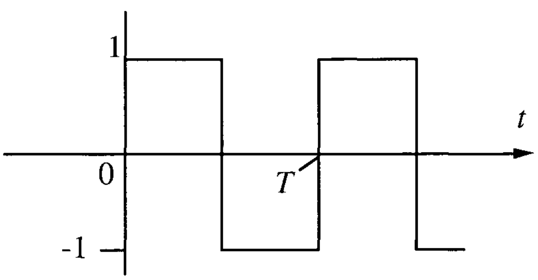
signs alternate, + for $n = 2$



p-phase rectified cosine wave ($p \geq 2$):

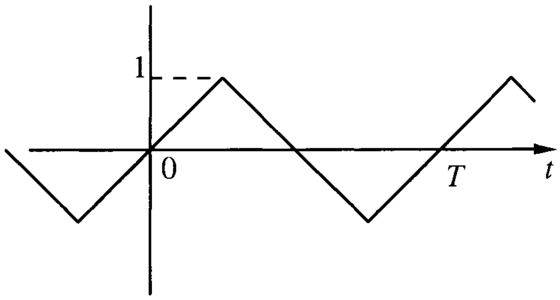
$$f(t) = \frac{p}{\pi}\sin\frac{\pi}{p}\left[1 + \frac{1}{\pi} \sum_{\substack{n=-\infty \\ n\ \mathrm{multiple} \\ \mathrm{of}\ p}}^{\infty} (\pm 1)\frac{e^{in\omega_0 t}}{n^2-1}\right]$$

signs alternate, + for $n = p$



Square wave:

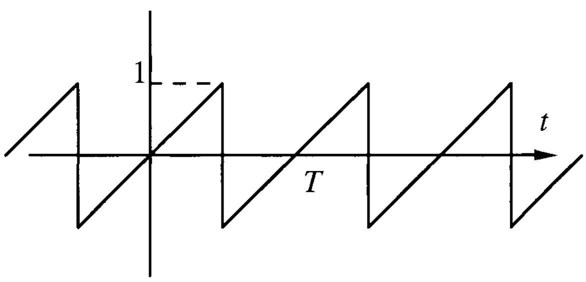
$$f(t) = \sum_{\substack{n=-\infty \\ n\ \mathrm{odd}}}^{\infty} \frac{2}{i\pi n}\, e^{in\omega_0 t}$$



Triangular wave:

$$f(t) = \frac{4}{i\pi^2} \sum_{\substack{n=-\infty \\ n\ \mathrm{odd}}}^{\infty} (\pm 1)\frac{e^{in\omega_0 t}}{n^2}$$

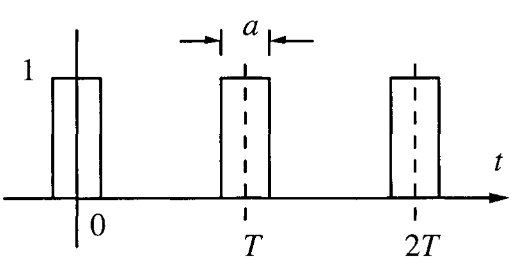
signs alternate, + for $n = 1$



Saw-tooth wave:

$$f(t) = \frac{1}{i\pi} \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} (\pm 1)\frac{e^{in\omega_0 t}}{n}$$

signs alternate, + for $n = 1$



Pulse wave:

$$f(t) = \frac{a}{T}\left[1 + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{\sin\frac{n\pi a}{T}}{\frac{n\pi a}{T}}\, e^{in\omega_0 t}\right]$$

Discrete Fourier Transform

The DFT of a sequence (x_n, $n = 0, 1, \ldots, N-1$) is defined by

$$X_k = \sum_{n=0}^{N-1} x_n \, e^{-i2\pi kn/N} \qquad \text{for } 0 \le k \le N-1$$

with inverse DFT

$$x_n = \frac{1}{N} \sum_{k=0}^{N-1} X_k \, e^{i2\pi kn/N} \qquad \text{for } 0 \le n \le N-1$$

Caution - Some books handle the $\frac{1}{N}$ factor differently and define transforms with differences in signs of the exponent

16. Laplace Transforms

$$\bar{x}(s) = \mathrm{L}\,(x(t)) = \int_{0-}^{\infty} x(t)\, e^{-st}\, dt$$

N.B. All functions in Laplace transform theory are <u>zero</u> for $t < 0$.

Initial Value Theorem:

If the limit as $s \to +\infty$ of $s\,\bar{x}(s)$ is finite, then

$$x(0^+) = \lim_{s \to +\infty} s\,\bar{x}(s)$$

Final Value Theorem:

Providing $x(t)$ tends to a limit as $t \to \infty$ then

$$\lim_{t \to \infty} x(t) = \lim_{s \to 0} s\,\bar{x}(s)$$

Table of Laplace Transforms

N.B. All functions in Laplace transform theory are <u>zero</u> for $t < 0$.

Function (for $t \geq 0$)	Transform	Remarks
$e^{-at}x(t)$	$\bar{x}(s+a)$	Shift in s
$x(t-\tau)\ H(t-\tau)$	$e^{-s\tau}\bar{x}(s)$	Shift in t $\tau \geq 0$
$\frac{dx(t)}{dt} \equiv x'(t)$	$s\bar{x}(s) - x(0)$	Differentiation
$\frac{d^2x(t)}{dt^2} \equiv x''(t)$	$s^2\bar{x}(s) - sx(0) - x'(0)$	
$\frac{d^nx(t)}{dt^n} \equiv x^{(n)}(t)$	$s^n\bar{x}(s) - s^{n-1}x(0) - s^{n-2}x'(0) - \ldots - sx^{(n-2)}(0) - x^{(n-1)}(0)$	
$\int_0^t x(\tau)d\tau$	$s^{-1}\ \bar{x}(s)$	Integration
$\int_0^t x_1(\tau)x_2(t-\tau)\,d\tau$	$\bar{x}_1(s)\ \bar{x}_2(s)$	Convolution
$t\,x(t)$	$-\frac{d}{ds}\bar{x}(s)$	
$1 \equiv H(t) \equiv u(t)$	s^{-1}	Heaviside step function
$\delta(t)$	1	Dirac delta function
$H(t-\tau)$	$s^{-1}e^{-s\tau}$	$\tau \geq 0$
$\delta(t-\tau)$	$e^{-s\tau}$	$\tau \geq 0$

Function (for $t \geq 0$)	Transform	Function (for $t \geq 0$)	Transform
t	s^{-2}	t^n	$n!\ s^{-n-1}$
e^{-at}	$(s+a)^{-1}$	$t^n e^{-at}$	$\frac{n!}{(s+a)^{n+1}}$
$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$	$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
$e^{-at}\sin \omega t$	$\frac{\omega}{(s+a)^2+\omega^2}$	$e^{-at}\cos \omega t$	$\frac{(s+a)}{(s+a)^2+\omega^2}$
$t\sin \omega t$	$\frac{2s\omega}{(s^2+\omega^2)^2}$	$t\cos \omega t$	$\frac{s^2-\omega^2}{(s^2+\omega^2)^2}$
$\sinh \omega t$	$\frac{\omega}{s^2-\omega^2}$	$\cosh \omega t$	$\frac{s}{s^2-\omega^2}$

17. Numerical Analysis

Finding roots of equations

Simple iteration

A method which sometimes works for an equation of the form $x = f(x)$ is to iterate

$$x_{n+1} = f(x_n)$$

Newton-Raphson

If the equation is $y = f(x)$ and x_n is an approximation to a root then a usually better approximation x_{n+1} is given by

$$x_{n+1} = x_n - f(x_n)\,/\,f'(x_n)$$

Numerical evaluation of Integrals

Trapezium Rule

$$\int_a^{a+h} y\,dx \approx \frac{h}{2}[\,y(a+h) + y(a)\,]$$

Thus, if the interval (a,b) is divided using n equal intervals each of length h,

$$\int_a^b y\,dx \approx \frac{h}{2}[\,y_0 + 2\,(\,y_1 + y_2 + \ldots + y_{n-1}\,) + y_n\,]$$

Simpson's Rule

$$\int_a^{a+2h} y\,dx \approx \frac{h}{3}[\,y(a+2h) + 4\,y(a+h) + y(a)\,]$$

Thus if the interval (a,b) is divided using n equal intervals, each of length h, with n even

$$\int_a^b y\,dx \approx \frac{h}{3}[\,y_0 + 4\,(\,y_1 + y_3 + \ldots + y_{n-1}\,) + 2\,(\,y_2 + y_4 + \ldots + y_{n-2}\,) + y_n\,]$$

Finite differences

One-sided:
$$\frac{u^{n+1}-u^n}{\Delta t} = \frac{1}{\Delta t}\left[\left\{u^n + \frac{du}{dt}\Delta t + \frac{d^2u}{dt^2}\frac{\Delta t^2}{2!} + \ldots\right\} - u^n\right] = \frac{du}{dt} + \frac{d^2u}{dt^2}\frac{\Delta t}{2} + \ldots$$

Centred:
$$\frac{u^{n+1}-u^{n-1}}{2\Delta t} = \frac{1}{2\Delta t}\left[\left\{u^n + \frac{du}{dt}\Delta t + \frac{d^2u}{dt^2}\frac{\Delta t^2}{2!} + \frac{d^3u}{dt^3}\frac{\Delta t^3}{3!} + \ldots\right\} - \left\{u^n - \frac{du}{dt}\Delta t + \frac{d^2u}{dt^2}\frac{\Delta t^2}{2!} - \frac{d^3u}{dt^3}\frac{\Delta t^3}{3!} + \ldots\right\}\right]$$
$$= \frac{du}{dt} + \frac{d^3u}{dt^3}\frac{\Delta t^2}{6} + \ldots$$

Integration of the generic ODE $\frac{du}{dt} = f(u,t)$

"Forward Euler"
$$\frac{u^{n+1}-u^n}{\Delta t} = f(u^n,t^n) = f^n$$

"Predictor-Corrector Method"

(i) $$u^* = u^n + \Delta t\, f(u^n,t^n)$$

(ii) $$u^{n+1} = u^n + \frac{\Delta t}{2}\left[f(u^n,t^n) + f(u^*,t^{n+1})\right]$$

"Fourth-order Runge Kutta method"

$$u^{n+1} = u^n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

where
$$k_1 = \Delta t\, f(u^n,t^n)$$
$$k_2 = \Delta t\, f\left(u^n + \frac{k_1}{2},\, t^n + \frac{\Delta t}{2}\right)$$
$$k_3 = \Delta t\, f\left(u^n + \frac{k_2}{2},\, t^n + \frac{\Delta t}{2}\right)$$
$$k_4 = \Delta t\, f(u^n + k_3,\, t^n + \Delta t)$$

Least-squares curve fitting

Straight line : $y = a + bx$

$$\begin{cases} an + b\sum_i x_i = \sum_i y_i \\ a\sum_i x_i + b\sum_i x_i^2 = \sum_i x_i y_i \end{cases}$$

Quadratic: $y = a + bx + cx^2$

$$\begin{cases} an + b\sum_i x_i + c\sum_i x_i^2 = \sum_i y_i \\ a\sum_i x_i + b\sum_i x_i^2 + c\sum_i x_i^3 = \sum_i x_i y_i \\ a\sum_i x_i^2 + b\sum_i x_i^3 + c\sum_i x_i^4 = \sum_i x_i^2 y_i \end{cases}$$

The cubic Ferguson Curve

$$\mathbf{r}(t) = \mathbf{p}(0)\left\{1 - 3t^2 + 2t^3\right\} + \mathbf{p}(1)\left\{3t^2 - 2t^3\right\} + \dot{\mathbf{p}}(0)\left\{t - 2t^2 + t^3\right\} + \dot{\mathbf{p}}(1)\left\{-t^2 + t^3\right\}$$

The cubic Bezier Curve

$$\mathbf{r}(t) = (1 - t)^3\mathbf{p}(0) + 3t(1 - t)^2\mathbf{p}(1) + 3t^2(1 - t)\,\mathbf{p}(2) + t^3\mathbf{p}(3)$$

18. Probability and Statistics

Discrete Random Variables

The probability that a random variable X takes the value r is denoted $P(X = r)$ or p_r. The mean, or expected value, of X is denoted E[X] and its variance Var[X]. The function $g(z)$ is said to be a generating function for X if,

$$g(z) = \sum_{\text{all } r} p_r z^r$$

With this definition: $\quad E[X] = \mu = g'(1)$

$$\mathrm{Var}[X] = \sigma^2 = E[X^2] - \mu^2 = g''(1) + g'(1) - g'(1)^2$$

Distribution	**Parameters ($q = 1 - p$)**	**$P(X=r) = p_r$**	**$g(z)$**	**E[X]**	**Var[X]**
Bernoulli	$0 < p < 1$	$p^r q^{1-r}$, $r = 0,1$	$q + pz$	p	pq
Binomial	n, $0 < p < 1$	$\binom{n}{r} p^r q^{n-r}$, $r = 0 \ldots n$	$(q + pz)^n$	np	npq
Geometric	$0 < p < 1$	$q^r p$, $r = 0 \ldots \infty$	$\dfrac{p}{1 - qz}$	$\dfrac{q}{p}$	$\dfrac{q}{p^2}$
Poisson	$\lambda > 0$	$e^{-\lambda} \dfrac{\lambda^r}{r!}$, $r = 0 \ldots \infty$	$e^{\lambda(z-1)}$	λ	λ

Continuous Random Variables

The probability that a random variable X takes a value in the range $(x, x + dx)$ is denoted $f(x)\,dx$. The cumulative probability function $P(X \leq x)$ is denoted $F(x)$. The mean or expected value of X is denoted E[X] and its variance Var[X]. The function $g(s)$ is said to be a generating function for X if,

$$g(s) = \int_{\text{all } x} e^{-sx} f(x)\,dx$$

With this definition: $\quad E[X] = \mu = -g'(0)$

$$\mathrm{Var}[X] = \sigma^2 = E[X^2] - \mu^2 = g''(0) - g'(0)^2$$

Distribution	**Params**	**f(x)**	**g(s)**	**E[X]**	**Var[X]**
Uniform	$a < b$	$\frac{1}{b-a}$ $a \le x \le b$ 0 otherwise	$\frac{e^{-as} - e^{-bs}}{s(b-a)}$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$
Exponential	$\lambda > 0$	$\lambda e^{-\lambda x}$ $x \ge 0$ 0 otherwise	$\frac{\lambda}{\lambda + s}$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$
Normal or Gaussian	$\sigma > 0$	$\frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left[\frac{x-\mu}{\sigma}\right]^2\right\}$ $-\infty < x < \infty$	$\exp(-s\mu + \frac{1}{2}s^2\sigma^2)$	μ	σ^2
Standard Normal		$\frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}x^2\right\}$ $-\infty < x < \infty$	$\exp(\frac{1}{2}s^2)$	0	1
Erlang-k	$k > 0$ $\mu > 0$	$\frac{\mu k(\mu k x)^{k-1}}{(k-1)!} e^{-\mu k x}$ $x \ge 0$ 0 otherwise	$\left(\frac{k\mu}{k\mu + s}\right)^k$	$\frac{1}{\mu}$	$\frac{1}{k\mu^2}$

Standard Normal Distribution

If X has a normal distribution with mean μ and standard deviation σ (denoted X ~ N(μ,σ)), then $Y = \frac{X-\mu}{s}$ has a normal distribution with mean 0 and standard deviation 1 (i.e. Y ~ N(0,1))
N(0,1) is referred to as the **standard** normal distribution.

Tables of the cumulative probability function $F(z) = \frac{1}{\sqrt{2\pi}} \int_{x=-\infty}^{z} \exp\left\{-\frac{1}{2}x^2\right\} dx$ for the standard normal distribution, which is usually denoted $\Phi(z)$, appear opposite.

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998

z	1.282	1.645	1.960	2.326	2.576	3.090	3.291	3.891	4.417
$\Phi(z)$	.90	.95	.975	.99	.995	.999	.9995	.99995	.999995
$2(1-\Phi(z))$	.20	.10	.05	.02	.01	.002	.001	.0001	.00001