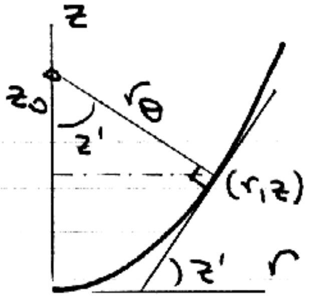
1. $2cz = r^2 \quad \therefore \quad cz' = r \quad ; \quad cz'' = 1 \quad ; \quad ' \equiv d/dr$




Equation of normal through $(r, z)$

$$z' = r/(z_0 - z) \quad \text{i.e.} \quad z_0 = z + r/z'$$

$\therefore$ Length of normal, $r_\theta$, is given by

$$r_\theta^2 = (z_0 - z)^2 + r^2 = r^2[1 + 1/(z')^2] = r^2 + c^2$$

& $r_\theta = (r^2 + c^2)^{1/2}$

The radius of curvature, $r_\phi$, is

$$r_\phi = [1 + (z')^2]^{3/2}/z'' \quad \text{and from above:}$$

$$= [1 + (r/c)^2]^{3/2}\, c = (r^2 + c^2)^{3/2}/c^2$$

Substitute into membrane stress equations (2):-

$$\sigma_\phi = (p/2t)\, r_\theta = k\,(r^2 + c^2)^{1/2} \quad ; \quad k = p/2t$$

$$\sigma_\theta = \sigma_\phi\,[2 - r_\theta/r_\phi] = k\,(r^2 + c^2)^{1/2}\,[\,2 - c^2/(r^2 + c^2)]$$

$$= k\,(2r^2 + c^2)/(r^2 + c^2)^{1/2} \qquad \text{QED.}$$

For the design of the glass dome, fairly obviously the membrane stresses are maximum when $r$ is maximum. Inserting values, assuming pressure and stresses in MPa and all dimensions in mm

$$\left.\begin{aligned} \sigma_\phi &= k\,(150^2 + 60^2)^{1/2} = 161.6\,k \\ \sigma_\theta &= k\,(2 \times 150^2 + 60^2)/161.6 = 300.8\,k \end{aligned}\right\} \text{ so the hoop stress } \sigma_\theta \text{ is greater}$$

Setting $\sigma_\theta = 35$ MPa, the design stress

$$p = \rho g h = 10\,(\text{kN/m}^3) \times 200\,(\text{m}) = 2\text{ MPa}$$

then $\quad 35 = 300.8\,p/2t \quad ; \quad t = 300.8 \times 2/2 \times 35 = \underline{\underline{8.6}}$ mm

2(a) $\sigma_\phi$ may be found from free body sketched, then $\sigma_\theta$ follows from membrane equation.

For the free body, $\quad p = \rho\,(h - z)$

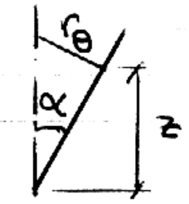
$$\therefore \quad \sigma_\phi \cos\alpha \,.\, t \,.\, 2\pi r = pA + W = pA + \rho\,\tfrac{1}{3} A z \quad \text{(vol of cone)}$$

where $A = \pi r^2$ and $r = z \tan\alpha$.

$$\therefore \quad 2\pi r\, t \cos\alpha\, \sigma_\phi = A\,[\,\rho(h - z) + \tfrac{1}{3}\rho z] = \pi r^2 \rho\,(3h - 2z)/6$$

$$\text{i.e.} \quad \sigma_\phi = k\,(3h - 2z)\,z \quad ; \quad k = \rho \tan\alpha / 6t \cos\alpha$$

Now require the radii $r_\theta$, $r_\phi$ for membrane equation. The radius of curvature of the meridian $\to \infty$ i.e. $r_\phi \to \infty$. From the sketch, $r_\theta = z \tan\alpha \sec\alpha$




So the membrane equation (ii) becomes :-

$$\sigma_\theta = r_\theta\,[\,p/t - \cancel{\sigma_\phi/r_\phi}\,] = (p/t)\, r_\theta \quad ; p = \rho(h - z)$$

$$= \rho\,(h - z)\,\tfrac{1}{t}\,.\, z \sin\alpha/\cos^2\alpha = k\,6z\,(h - z)$$

QED.

(b)

Putting the membrane equations & $\sigma_3 = 0$ into dist$^n$ equation

$$2\sigma_E^2 = (\sigma_\theta - 0)^2 + (\sigma_\phi - 0)^2 + (\sigma_\theta - \sigma_\phi)^2$$

$$2(\sigma_E/kz)^2 = 36(h - z)^2 + (3h - 2z)^2 + [\,6h - 6z - 3h + 2z\,]^2$$

$$\sigma_E^2 = (kz)^2\,[\,27h^2 - 54hz + 28z^2\,]$$

Differentiating for max $\sigma_E$ :-

$$d(\sigma_E^2)/dz = 0 = 56z^2 - 81hz + 27h^2 \qquad ; h = 6.5\text{ m}$$

$$\therefore \quad z = 6.015\text{ m} \quad \text{or} \quad 3.386\text{ m}$$

2 (concl'd) It may easily be shown that the latter smaller value of $z$ corresponds to max $\sigma_E$, i.e. $\sigma^*$. So

$$\sigma^*/k = 3.386\,[27 \times 6.5^2 - 54 \times 6.5 \times 3.386 + 28 \times 3.386^2]^{1/2} = 56.0\ \mathrm{m}^2$$

$$\underline{t} = \sigma^*\, 6t \cos\alpha / \rho \tan\alpha \quad ; \quad \rho = 9.81\ \mathrm{kN/m^3}.$$

$$\therefore\ t = \underset{\mathrm{m^2}}{56.0} \times \underset{\mathrm{N/m^3}}{9.81 \times 10^3} \times 1 / 6 \times \underset{\mathrm{m^2/N}}{65 \times 10^6} \times 1/\sqrt{2} = 1.99\ \mathrm{mm} \qquad \text{say } \underline{2\ \mathrm{mm}}$$

At water level, $z = h$, so membrane stresses from above

$$\sigma_\theta = 0 \quad ; \quad \sigma_\phi = k h^2$$

$$\therefore\ \epsilon_D = \epsilon_\theta = (\cancel{\sigma_\theta} - \nu \sigma_\phi)/E = -\nu \rho h^2 \tan\alpha / 6 E t \cos\alpha$$

$$= -0.3 \times \underset{\mathrm{N/m^3}}{9.81 \times 10^3} \times \underset{\mathrm{m^2}}{6.5^2} \times 1 / 6 \times \underset{\mathrm{m^2/N}}{207 \times 10^9} \times \underset{1/\mathrm{m}}{2 \times 10^{-3}} \times 1/\sqrt{2}$$

$$= -71 \times 10^{-6}$$ i.e. compression of 71 $\mu$strain.