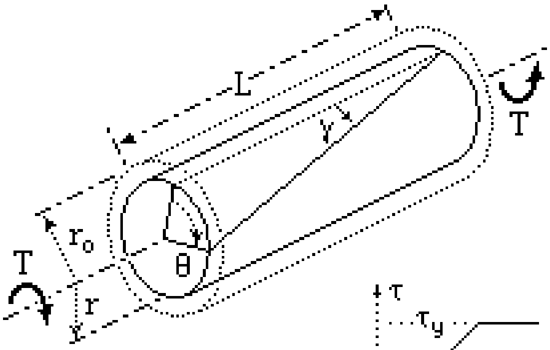
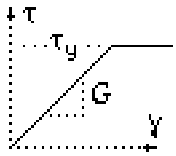
APPENDIX : PRESETTING A TORSION BAR



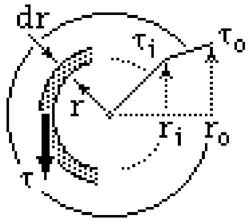
We shall first derive some general results for a homogeneous solid torsion bar of radius r_o and length L, made from an elastic-perfectly plastic material whose shear yield and modulus are τ_y (ie S_{ys} above) and G respectively.

When torque T is applied to the ends of the bar, one end rotates through an angle θ with respect to the other. Cross-sections remain plane and any radial line in a cross-section remains straight, no matter whether behaviour is elastic or plastic. From the sketch, compatibility requires the shear strain γ at radius r to satisfy :



$$L\gamma = r\theta \quad \text{or}$$

(i) $\gamma = \theta'.r$ where $\theta' \equiv \theta / L$ is constant for all radii in the cross-section.



From the constitutive law, the corresponding stress is :

(ii) $\tau = G\gamma = G\theta'.r$ from (i) if elastic, or
$= \tau_y$ if plastic; ie $\tau = \text{minimum}\,(G\theta'.r,\ \tau_y)$

The torque T is the resultant of shear stress over the cross-section, ie from the sketch :

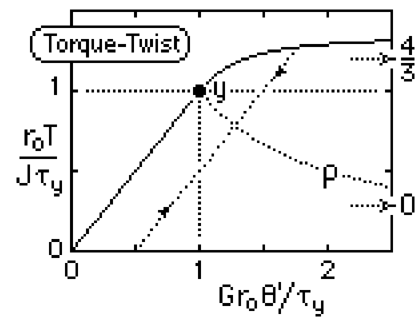
$$T = \int_0^{r_o} (2\pi r.dr)\,\tau\,.r = 2\pi \int_0^{r_o} \tau\, r^2\, dr \quad \text{where } \tau = f(r).$$

A piece-wise linear variation is defined by a known stress τ_i at some interior radius r_i, together with a stress τ_o at the outer surface, radius r_o. From the foregoing, the torque which corresponds to this stress pattern is :

$$T = 2\pi \int_0^{r_i} (\tau_i\, {}^{r}/_{r_i})\, r^2\, dr + 2\pi \int_{r_i}^{r_o} \{\tau_i + (\tau_o - \tau_i).(r - r_i)/(r_o - r_i)\}\, r^2\, dr \quad \text{ie}$$

(iii) $T r_o / J = \tau_o + (\tau_i - \rho\tau_o)(1-\rho^3) / 3(1-\rho)$ in which $\rho \equiv {}^{r_i}/_{r_o}$ and $J = {}^{\pi}/_2\, r_o^4$

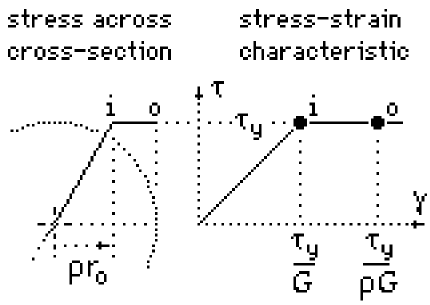
The above results will now be applied to presetting and loading of the torsion bar, which involve the following steps.



Under increasing load the torsion bar responds elastically until the maximum stress (at the outer surface, radius r_o) reaches the yield stress, τ_y. The linear portion of the bar's normalised torque-twist characteristic is terminated by the point **y** where the torque T_y and the twist θ_y' correspond to τ_y occurring at r_o. From the above, or from the familiar elastic equations, these critical parameters are related through :

(iv) $\tau_y / r_o = T_y / J = G\,\theta_y'$

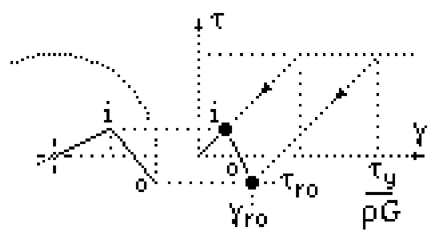
The torque and angle of twist are further increased with progressive yielding of the bar's outer skin and shrinking of its elastic core. At any particular load the stress in the core varies linearly from zero at the axis to τ_y at $r_i = \rho r_o$, the elastic-plastic interface, while the stress in the skin is constant at yield. From (ii) applied to yield at the interface, and from (iii) with $\tau_i = \tau_o = \tau_y$, the variations of torque T and twist $\theta' \geq \theta_y'$ with core size ρ are :



(v) $r_o T / J\tau_y = {}^1/_3\,(4 - \rho^3)$; $G r_o \theta' / \tau_y = {}^1/\rho$

which results in the non-linear portion of the torque-twist characteristic.

This completes the loading phase of the presetting operation, but if loading were to continue then the limit load T_p would be reached when plasticity had extended right throughout the bar cross-section - whereupon, from (v) with $\rho = 0$: $r_o T_p / J\tau_y = {}^4/_3$. That is the plastic collapse torque T_p is ${}^4/_3$ the torque at which yielding first commences, T_y.



The preload T of the previous step which caused yielding of the skin to radius $r_i = \rho r_o$ is now removed completely. All the material, including that which has reached yield, unloads elastically as suggested by the straight unloading line of the torque-twist characteristic above. However although there is zero net torque after unloading, residual stresses and a permanent set have been induced, as sketched here.

To find this permanent twist, θ_r', together with the residual stresses τ_{ri} and τ_{ro}, we note the compatibility requirements (i) : $\gamma_{ri} = r_i\,\theta_r'$ and $\gamma_{ro} = r_o\,\theta_r'$. Since the interface lies on the elastic line, $\tau_{ri} = G\gamma_{ri}$, and considering the linear unloading process at the outer surface from the previous stress/strain state above, $\tau_{ro} = \tau_y - G\,(\tau_y / \rho G - \gamma_{ro})$.
From these and from (iii) with zero net torque we obtain :

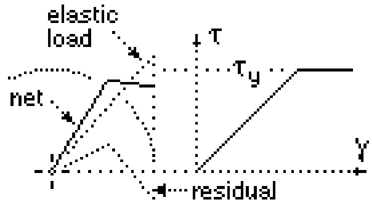
(vi) $\quad G r_o \theta_r' / \tau_y = (3 - 4\rho + \rho^4)/3\rho$; $\quad \tau_{ri}/\tau_y = (3 - 4\rho + \rho^4)/3$; $\quad \tau_{ro}/\tau_y = (\rho^3 - 1)/3 \le 0$

Plotting these states above demonstrates the negative shear on the outer surface which will go some way towards counteracting any subsequent positive shear due to an applied load.

The bar, having been preset, is now put into operation where it sustains the service torque T_s. Assuming the bar to behave elastically still (we shall check this in a minute), the stress distribution will be a superposition of the elastic load stress $\tau / r = T_s / J$, and the residual stress given by (vi) whose torque resultant is zero. Thus :

(vii) $\quad \tau_o = r_o T_s / J + (\rho^3 - 1)\,\tau_y / 3$

$\quad \tau_i = \rho . r_o T_s / J + (3 - 4\rho + \rho^4)\,\tau_y / 3$ which are plotted here.



The maximum service torque $T_{s.hi}$ will be that value of T_s which causes τ_o or τ_i to just reach yield. This is easily found from (vii) to be :

(viii) $\quad r_o T_{s.hi} / J\tau_y = (4 - \rho^3)/3$ - exactly the same result as (ii).

It is evident therefore that this ***elastic*** torque capacity of the preset bar is the same as the ***plastic*** torque which was used to preset the bar initially, and is higher than the torque which just started to yield the bar, $r_o T_y / J\tau_y = 1$ from (iv). Thus loading after presetting involves elastic return up the unloading line shown in the torque-twist characteristic above.

Presetting thus increases the torque capacity of a bar by decreasing the stresses in the highly stressed outer skin, whilst compensating by an increase of stress level in the lightly loaded core.